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The Interaction of a Quantum Mechanical Oscillator with Gravitational Radiation

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Abstract

Within the framework of the linearized field equations of gravitation, the interaction operators between a quantum mechanical system and an external gravitational field are derived from the general-covariant Klein–Gordon and Dirac equation. In the case of linearly polarized plane gravitational waves the transition probabilities for absorption and induced and spontaneous emission of gravitational radiation by a quantum mechanical harmonic oscillator are calculated with the help of the time-dependent perturbation method. The results coincide with the classical ones according to the correspondence principle.

1. *Introduction*

Most previous investigations of the interaction of gravitational waves with matter have been performed within the framework of classical mechanics (cf., e.g., Weber, 1960; Frehland, 1971; Papapetrou, 1972). On the other hand, it is to be expected that especially at low temperatures quantum mechanical effects play a role in the excitation of a material system by external gravitational waves. Therefore in this work we will discuss the interaction of gravitational waves with a quantum mechanical system. Certainly for high excitation states the results should correspond to the classical ones asymptotically, so that these get a further foundation by the following investigation also. The gravitational waves are treated within the linearized field equations of Einstein's theory of gravitation; as quantum mechanical system we choose an *ideal* harmonic oscillator, the internal potential of which will not be modified by the external gravitational fields.

At first the interaction operator between a quantum mechanical system and a linearized gravitational field will be derived from the general-covariant Klein–Gordon and Dirac theories, respectively. Subsequently we go over to the nonrelativistic Schrödinger equation and calculate the excitation of an ideal linear harmonic oscillator by a plane linearly polarized gravitational wave with the use of the time-dependent perturbation method. This may be justified in view of the fact that the deviations of the metric from that of the flat space-time will be considered as small.

In this way the transition probabilities for *absorption* and *induced emission* of gravitational radiation by an oscillator are obtained including the selection rules. From these the transition probabilities for *spontaneous emission* of gravitational radiation can be deduced by a consideration analogous to that of Einstein for the electromagnetic case. We find that the transition probabilities are determined by the matrix elements of the mass-quadrupole operator of the oscillator and that the selection rule for the quantum number n is given by $\Delta n = \pm 2$ corresponding to the quadrupole interaction of the gravitational radiation. According to this the *total* gravitational radiation energy absorbed by an oscillator with time will increase linearly with the quantum number n. Thus a highly excited oscillator seems to be more appropriate for detection of (in general weak) gravitational radiation than oscillators in the ground and the lower excited states.

2. *Interaction Operator According to the Klein–Gordon Theory*

We start from the general-covariant Klein–Gordon equation (cf., e.g., Schmutzer, 1968), which takes in case of the signature + 2 for the metric of space-time the form[1]

$$\Psi^{|\mu}{}_{\|\mu} - 2i(e/\hbar c)A_\mu \Psi^{|\mu} - i(e/\hbar c)A^\mu{}_{\|\mu}\Psi - (e/\hbar c)^2 A_\mu A^\mu \Psi - (mc/\hbar)^2 \Psi = 0 \tag{2.1}$$

Here A_μ is the electromagnetic four-potential, to which the particle described by the wave function Ψ is exposed (m is the rest mass and e is the charge of the particle). With the use of the Lorentz convention

$$A^\mu{}_{\|\mu} = 0 \tag{2.1a}$$

Equation (2.1) simplifies to

$$(g^{\mu\nu}\Psi_{|\mu})_{|\nu} + g^{\mu\lambda}\Gamma^\nu_{\mu\nu}\Psi_{|\lambda} - 2i(e/\hbar c)g^{\mu\nu}A_\mu \Psi_{|\nu} - (e/\hbar c)^2 A_\mu A^\mu \Psi - (mc/\hbar)^2 \Psi = 0 \tag{2.2}$$

where the covariant derivative is written out. Insertion of the definition of the Christoffel symbols $\Gamma^\nu_{\mu\lambda}$ and execution of the derivative of the first term yields

$$g^{\mu\nu}\Psi_{|\mu|\nu} + (\tfrac{1}{2}g_{\nu\sigma|\mu} - g_{\sigma\mu|\nu})g^{\nu\sigma}g^{\mu\lambda}\Psi_{|\lambda} - 2i(e/\hbar c)g^{\mu\nu}A_\mu\Psi_{|\nu} - (e/\hbar c)^2 g^{\mu\nu}A_\mu A_\nu \Psi - (mc/\hbar)^2\Psi = 0 \tag{2.3}$$

[1] $\|\nu$ signifies the covariant and $|\nu$ the ordinary partial derivative with respect to the coordinate x^ν.

Furthermore we use the metric of space-time in linear approximation with respect to the flat one[2]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\alpha\beta} = \eta^{\alpha\beta} + \gamma^{\alpha\beta}$$

$$|h_{\mu\nu}| \ll 1, \qquad |\gamma^{\alpha\beta}| \ll 1; \qquad \eta_{\mu\nu}, \eta^{\alpha\beta} = \begin{pmatrix} +1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & -1 \end{pmatrix} \tag{2.4}$$

Lifting and lowering of tensor indices will be performed in the following by $\eta^{\alpha\beta}$ and $\eta_{\mu\nu}$, respectively. Then

$$\gamma^{\alpha\beta} = -h^{\alpha\beta} \tag{2.4a}$$

is valid. Additionally we demand the de Donder condition given by

$$h_\mu{}^\alpha{}_{|\alpha} - \tfrac{1}{2} h_{|\mu} = 0, \qquad (h = h_\alpha{}^\alpha) \tag{2.4b}$$

According to (2.4b) in the linear approximation the second term in (2.3) vanishes and we obtain, with the use of (2.4) and (2.4a),

$$\eta^{\mu\nu}\Psi_{|\mu|\nu} - h^{\mu\nu}\Psi_{|\mu|\nu} - 2i(e/\hbar c)\eta^{\mu\nu}A_\mu\Psi_{|\nu} + 2i(e/\hbar c)h^{\mu\nu}A_\mu\Psi_{|\nu}$$
$$- (e/\hbar c)^2\eta^{\mu\nu}A_\mu A_\nu\Psi + (e/\hbar c)^2 h^{\mu\nu}A_\mu A_\nu\Psi - (mc/\hbar)^2\Psi = 0 \tag{2.5}$$

With the magnetic vector potential $\mathbf{A} = \{A_a\}$ and the electric potential energy $V = -eA_4$, Eq. (2.5) can be written

$$\Delta\Psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi - 2i\frac{e}{\hbar c}\mathbf{A}\,\mathrm{grad}\,\Psi - 2i\frac{V}{\hbar c^2}\frac{\partial\Psi}{\partial t} - \left(\frac{e}{\hbar c}\right)^2\mathbf{A}^2\Psi + \left(\frac{V}{\hbar c}\right)^2\Psi$$
$$- \left(\frac{mc}{\hbar}\right)^2\Psi + G = 0 \tag{2.6}$$

where the abbreviation

$$G = -h^{\mu\nu}\Psi_{|\mu|\nu} + 2i(e/\hbar c)h^{\mu\nu}A_\mu\Psi_{|\nu} + (e/\hbar c)^2 h^{\mu\nu}A_\mu A_\nu\Psi \tag{2.6a}$$

represents the deviation of Eq. (2.6) from the Klein-Gordon equation in flat space-time in consequence of the interaction with the gravitational field. Splitting the terms of (2.6a) in view of the derivatives with respect to space and to time we find

$$G = -h^{ab}\Psi_{|a|b} - 2h^{a4}\Psi_{|a|4} - h^{44}\Psi_{|4|4} + 2i(e/\hbar c)h^{\mu a}A_\mu\Psi_{|a}$$
$$+ 2i(e/\hbar c)h^{\mu 4}A_\mu\Psi_{|4} + (e/\hbar c)^2 h^{\mu\nu}A_\mu A_\nu\Psi \tag{2.6b}$$

wherein Latin indices, as in the following, run from 1 to 3. The relation (2.6b) represents the interaction operator with the gravitational field applied to the wave function Ψ in the framework of the Klein-Gordon theory.

[2] We choose the coordinate $x^4 = ct$.

3. *Transition to the Schrödinger Theory*

For calculation of the transition probabilities in the case of excitation by gravitational waves we restrict ourselves to the nonrelativistic quantum theory. The nonrelativistic limit of Eq. (2.6) will be obtained by neglecting the terms proportional to negative powers of the velocity of light. Furthermore, it is to be taken into account that the energy in the relativistic theory is greater than in the nonrelativistic one by the amount of the rest energy mc^2; therefore we choose the usual relation between the relativistic wave function Ψ and the Schrödinger wave function Φ:

$$\Psi(x^\mu) = e^{-i(mc^2/\hbar)t}\Phi(x^\mu) \tag{3.1}$$

Herewith we get from Eqs. (2.6) and (2.6b) at first:

$$\begin{aligned}\Delta\Phi &+ 2i\frac{m}{\hbar}\frac{\partial\Phi}{\partial t} - 2\frac{m}{\hbar^2}V\Phi - 2i\frac{e}{\hbar c}\mathbf{A}\,\mathrm{grad}\,\Phi - \frac{\partial^2\Phi}{c^2\partial t^2} - 2i\frac{V}{\hbar c^2}\frac{\partial\Phi}{\partial t} - \left(\frac{e}{\hbar c}\right)^2\mathbf{A}^2\Phi \\ &+ \left(\frac{V}{\hbar c}\right)^2\Phi + \left(\frac{mc}{\hbar}\right)^2 h^{44}\Phi + 2i\frac{mc}{\hbar}h^{a4}\Phi_{|a} - h^{ab}\Phi_{|a|b} + 2i\frac{m}{\hbar}h^{44}\frac{\partial\Phi}{\partial t} \\ &+ 2\frac{em}{\hbar^2}h^{\mu 4}A_\mu\Phi - 2h^{a4}\frac{\partial}{c\partial t}\Phi_{|a} + 2i\frac{e}{\hbar c}h^{\mu a}A_\mu\Phi_{|a} - h^{44}\frac{\partial^2\Phi}{c^2\partial t^2} \\ &+ 2i\frac{e}{\hbar c^2}h^{\mu 4}A_\mu\frac{\partial\Phi}{\partial t} + \left(\frac{e}{\hbar c}\right)^2 h^{\mu\nu}A_\mu A_\nu\Phi = 0\end{aligned} \tag{3.2}$$

Neglecting in (3.2) all terms up to the explicit order of c^{-2} we obtain the generalized Schrödinger equation:

$$\begin{aligned}\frac{\hbar^2}{2m}\Delta\Phi &+ i\hbar\frac{\partial\Phi}{\partial t} - V\Phi - i\frac{e\hbar}{mc}\mathbf{A}\,\mathrm{grad}\,\Phi + \tfrac{1}{2}mc^2h^{44}\Phi + i\hbar c h^{a4}\Phi_{|a} \\ &- \tfrac{1}{2}\frac{\hbar^2}{m}h^{ab}\Phi_{|a|b} + i\hbar h^{44}\frac{\partial\Phi}{\partial t} + eh^{\mu 4}A_\mu\Phi - \frac{\hbar^2}{mc}h^{a4}\frac{\partial}{\partial t}\Phi_{|a} \\ &+ i\frac{e\hbar}{mc}h^{\mu a}A_\mu\Phi_{|a} = 0\end{aligned} \tag{3.3}$$

where all the terms containing components of $h_{\mu\nu}$ result from the interaction with the gravitational field. It is of interest that the fifth term corresponds to the third one and the sixth term to the fourth one describing the usual interaction with the electromagnetic and the "gravitational" four-vector potential, respectively, whereas the remaining terms of the gravitational interaction originate from the tensorial character of the gravitational field and consequently do not possess an electromagnetic analogy. In the following we specify the gravitational interaction operator for several gravitational fields.

3.1. Interaction Operator for Stationary and Static Gravitational Fields

The *stationary* gravitational field is characterized (choosing suitable coordinates) by

$$h_{\mu\nu|4} = 0, \qquad h_{\mu\nu} = O(c^{-2}) \tag{3.4}$$

Herewith Eq. (3.3) takes the form, neglecting all the terms proportional to c^{-2} and smaller,

$$\frac{\hbar^2}{2m}\Delta\Phi + i\hbar\frac{\partial\Phi}{\partial t} - V\Phi - i\frac{e\hbar}{mc}\mathbf{A}\ \mathrm{grad}\ \Phi + \tfrac{1}{2}mc^2h^{44}\Phi + i\hbar ch^{a4}\Phi_{|a} = 0 \tag{3.5}$$

Accordingly in stationary fields the electromagnetic and gravitational interaction corresponds exactly to one another.

In the *static* Newtonian approximation of gravitational fields the following is valid:

$$h_{11} = h_{22} = h_{33} = h_{44} = -2\frac{\mathcal{U}}{c^2}, \qquad h_{\mu\nu} = 0 \text{ otherwise} \tag{3.6}$$

where $\mathcal{U}$ is the (negative) Newtonian gravitational potential. With (3.6) Eq. (3.5) results in the absence of magnetic fields ($\mathbf{A} = 0$) in

$$(\hbar^2/2m)\,\Delta\Phi - (V + m\mathcal{U})\,\Phi = -i\hbar\,\partial\Phi/\partial t \tag{3.7}$$

which is exactly the expected form of the time-dependent Schrödinger equation in the case of static Newtonian gravitational fields.[3]

3.2. Interaction Operator for Gravitational Wave Fields

In the case of linearized plane gravitational waves the metric takes the general form

$$h_{11}(u) = -h_{22}(u) \neq 0, \quad h_{12}(u) \neq 0, \qquad h_{\mu\nu} = 0 \text{ otherwise}; \qquad u = t - x^3/c \tag{3.8}$$

if the wave propagates in the direction of the x^3 coordinate. A corresponding representation of $h_{\mu\nu}$ is valid, if the propagation direction is determined by the x^1 or x^2 coordinate. Then the Schrödinger equation (3.3) simplifies to

$$\frac{\hbar^2}{2m}\Delta\Phi - V\Phi - i\frac{e\hbar}{mc}\mathbf{A}\ \mathrm{grad}\ \Phi - \frac{\hbar^2}{2m}h^{ab}\Phi_{|a|b} + i\frac{e\hbar}{mc}h^{ab}A_b\Phi_{|a} = -i\hbar\frac{\partial\Phi}{\partial t} \tag{3.9}$$

Accordingly in the absence of magnetic fields ($\mathbf{A} = 0$) the interaction operator with gravitational waves is given by

$$W = \frac{\hbar^2}{2m}h^{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b} \tag{3.10}$$

[3] It may be of interest, that even in the case of purely gravitational interaction ($V = 0$) the mass of the particle does not fall out in spite of the validity of the equivalence principle. Therefore the gravi-quantum-mechanical effects depend on the mass of the considered particles in general.

and the Schrödinger equation reads

$$\frac{\hbar^2}{2m}\Delta\Phi - V\Phi - \frac{\hbar^2}{2m}h^{ab}\Phi_{|a|b} = -i\hbar\frac{\partial\Phi}{\partial t} \tag{3.11}$$

4. *Interaction Operator According to the Dirac Theory*

Subsequently we will show that in case of gravitational waves the interaction operator (3.10) follows also from the covariant Dirac equation in the nonrelativistic limit. We start with the covariant Dirac equation (cf., e.g., Schmutzer, 1968; Nowotny, 1972):

$$\gamma^\mu[\Psi_{\|\mu} - i(e/\hbar c)A_\mu\Psi] - (mc/\hbar)\Psi = 0 \tag{4.1}$$

wherein

$$\Psi_{\|\mu} = \Psi_{|\mu} - \Gamma_\mu\Psi \tag{4.1a}$$

is the covariant spinor derivative with the spinorial connection coefficients

$$\Gamma_\mu = \tfrac{1}{4}h_{(\rho)}{}^{\nu}{}_{\|\mu}\, h_\lambda{}^{(\rho)}\gamma^\lambda\gamma_\nu \tag{4.1b}$$

defined by the use of the orthonormal tetrad field $h_{\mu(\rho)}$.[4] The generalized Dirac matrices γ^μ satisfying the relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ are given by

$$\gamma^\mu = h^\mu{}_{(\rho)}\gamma^{(\rho)} \tag{4.1c}$$

where $\gamma^{(\rho)}$ are Dirac's *standard* matrices.

Restricting ourselves to linearized gravitational plane waves propagating in the direction of the x^3 coordinate we take with respect to the metric (3.8) the following tetrad field:

$$\begin{aligned}
h_{(1)\mu} &= (1 + \tfrac{1}{2}h_{11}, \tfrac{1}{2}h_{12}, 0, 0); & h_{(1)}{}^\mu &= (1 - \tfrac{1}{2}h_{11}, -\tfrac{1}{2}h_{12}, 0, 0)\\
h_{(2)\mu} &= (\tfrac{1}{2}h_{12}, 1 - \tfrac{1}{2}h_{11}, 0, 0); & h_{(2)}{}^\mu &= (-\tfrac{1}{2}h_{12}, 1 + \tfrac{1}{2}h_{11}, 0, 0)\\
h_{(3)\mu} &= (0, 0, 1, 0); & h_{(3)}{}^\mu &= (0, 0, 1, 0)\\
h_{(4)\mu} &= (0, 0, 0, -1); & h_{(4)}{}^\mu &= (0, 0, 0, 1)
\end{aligned} \tag{4.2}$$

Herewith we get from (4.1b) with the use of (4.1c) and (3.8) by a simple but long calculation the result

$$\Gamma_\mu \equiv 0 \tag{4.3}$$

Consequently Eq. (4.1) takes with respect to (4.1a), (4.1c), and (4.2) the following form:

$$\begin{aligned}
&\gamma^{(a)}\Psi_{|a} - \tfrac{1}{2}h^a{}_b\gamma^{(b)}\Psi_{|a} - \gamma^{(4)}\Psi_{|4} - i(e/\hbar c)\gamma^{(a)}A_a\Psi + i(e/2\hbar c)h^a{}_b\gamma^{(b)}A_a\Psi\\
&\quad + i(e/\hbar c)\gamma^{(4)}A_4\Psi - (mc/\hbar)\Psi = 0
\end{aligned} \tag{4.4}$$

[4] The index within the bracket means the tetrad index. The orthonormal tetrad field $h_{\mu(\rho)}$ satisfies the completeness relations $h_{\mu(\lambda)}h^\mu{}_{(\rho)} = \eta_{(\lambda\rho)}$, $h_{\mu(\rho)}h_\nu{}^{(\rho)} = g_{\mu\nu}$.

wherein the spacelike and timelike terms are already separated. Neglecting magnetic fields ($A_a = 0$) we get from (4.4), after multiplication with $ic\hbar\gamma^{(4)}$ from the left-hand side,

$$\alpha^a p_a c\Psi - \tfrac{1}{2}ch^a{}_b\alpha^b p_a\Psi + V\Psi + mc^2\beta\Psi - i\hbar\partial\Psi/\partial t = 0 \tag{4.5}$$

with the abbreviations

$$\begin{aligned} \alpha^a &= \gamma^{(4)}\gamma^{(a)} \\ \beta &= i\gamma^{(4)} \\ p_a &= -i\hbar\, \partial/\partial x^a \end{aligned} \tag{4.5a}$$

and with $V = -eA_4$ as electric potential energy. Equation (4.5) represents the special-relativistic form of the Dirac equation expanded by the second term on the left-hand side, which describes the interaction of the gravitational wave with the spinor field Ψ.

For the transition to the nonrelativistic theory we set in analogy to (3.1)

$$\Psi(x^\mu) = e^{-i(mc^2/\hbar)t}\tilde{\Phi}(x^\mu) \tag{4.6}$$

Herewith Eq. (4.5) takes the form

$$\alpha^a p_a c\tilde{\Phi} - \tfrac{1}{2}ch^a{}_b\alpha^b p_a\tilde{\Phi} + V\tilde{\Phi} + mc^2\beta\tilde{\Phi} - i\hbar\, \partial\tilde{\Phi}/\partial t - mc^2\tilde{\Phi} = 0 \tag{4.7}$$

With the usual ansatz

$$\tilde{\Phi}(x^\mu) = \begin{pmatrix} \Phi(x^\mu) \\ X(x^\mu) \end{pmatrix} \tag{4.8a}$$

we obtain from (4.7) using the representation

$$\alpha^a = \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.8b}$$

(σ^a Pauli matrices) the following two second-rank spinor equations:

$$\sigma^a P_a c X + V\Phi = i\hbar\, \partial\Phi/\partial t \tag{4.9a}$$

$$\sigma^a P_a c\Phi + (V - 2mc^2)X = i\hbar\, \partial X/\partial t \tag{4.9b}$$

Here

$$P_a = (\delta_a{}^b - \tfrac{1}{2}h_a{}^b)p_b \tag{4.9c}$$

is the generalized momentum operator containing the interaction with the gravitational wave. In the nonrelativistic limit ($|V| \ll mc^2$, $|i\hbar\, \partial X/\partial t| \ll mc^2|X|$) Eq. (4.9b) gives

$$X = (\sigma^a P_a/2mc)\, \Phi \tag{4.10}$$

and herewith Eq. (4.9a) results in:

$$\frac{\sigma^a \sigma^b P_a P_b}{2m} \Phi + V\Phi = i\hbar\, \partial\Phi/\partial t \tag{4.11}$$

With the relation

$$\sigma^a \sigma^b = \eta^{ab} + i\epsilon^{abc}\sigma_c \tag{4.12}$$

(ϵ^{abc} is the permutation symbol) we get from (4.11)

$$(1/2m)(P^a P_a + i\epsilon^{abc}\sigma_c P_a P_b)\Phi + V\Phi = i\hbar\, \partial\Phi/\partial t \tag{4.13}$$

where in the linear approximation *with respect to the metric (3.8)* and the definition of P_a according to (4.9c) and (4.5a)

$$\epsilon^{abc}\sigma_c P_a P_b = \frac{\hbar^2}{c}\sigma_2 \left[\left(\frac{\partial}{\partial t} h_{11}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial}{\partial t} h_{12}\right)\frac{\partial}{\partial x^2}\right] - \frac{\hbar^2}{c}\sigma_1 \left[\left(\frac{\partial}{\partial t} h_{21}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial}{\partial t} h_{22}\right)\frac{\partial}{\partial x^2}\right] \tag{4.14a}$$

and

$$P^a P_a = p^a p_a - h^{ab} p_a p_b \tag{4.14b}$$

Evidently, the terms (4.14a) are proportional to c^{-1} in contrast to the terms (4.14b) and must therefore be neglected in the nonrelativistic limit of Eq. (4.13). Herewith Eq. (4.13) goes over into the generalized Schrödinger equation [using the definition of p_a according to (4.5)]:

$$\frac{\hbar^2}{2m}\Delta\Phi - V\Phi - \frac{\hbar^2}{2m} h^{ab}\Phi_{|a|b} = -i\hbar \frac{\partial\Phi}{\partial t} \tag{4.15}$$

This result is identical with Eq. (3.11). Thus also according to the Einstein-Dirac theory the interaction operator of a quantum mechanical system with a gravitational wave is in the nonrelativistic limit defined by

$$W = \frac{\hbar^2}{2m} h^{ab} \frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b} \tag{4.16}$$

in agreement with Eq. (3.10).

5. *Excitation of a Linear Harmonic Oscillator by Gravitational Waves*

Now the excitation of a linear harmonic quantum mechanical oscillator through a gravitational wave will be calculated by the usual time-dependent perturbation method. Hereby we regard the gravitational potentials $h_{\mu\nu}$ as small perturbations of the flat space-time, in which the harmonic oscillator possesses well defined orthonormal eigenstates

$$\Phi_k = \varphi_k(x^a)\, e^{-iE_k t/\hbar} \tag{5.1a}$$

with the energy eigenvalues E_k satisfying the undisturbed Schrödinger equation

$$(\hbar^2/2m)\Delta\Phi_k - V\Phi_k = -i\hbar\, \partial\Phi_k/\partial t \tag{5.1b}$$

The perturbation of the oscillator by the gravitational wave will be taken into account by expanding the perturbed wave function Φ with respect to Φ_k according to

$$\Phi = \sum_k a_k(t)\Phi_k \tag{5.2}$$

and by solving the perturbed Schrödinger equation (3.11) or (4.15) with the use of (5.2). The potential V of the oscillator shall not be influenced by the gravitational perturbation (*ideal* oscillator).

Thus the insertion of (5.2) into (3.11) or (4.15) yields, using the Eqs. (5.1) and the orthonormality of the eigenfunctions Φ_k,

$$i\hbar \frac{\partial a_k}{\partial t} = \sum_l W_{kl} a_l\, e^{i(E_k - E_l)t/\hbar} \tag{5.3}$$

wherein the matrix elements [cf. (4.16)]

$$W_{kl} = \int \varphi_k^* W \varphi_l\, d^3x = \frac{\hbar^2}{2m} \int \varphi_k^* h^{ab} \varphi_{l\,|\,a\,|\,b}\, d^3x \tag{5.3a}$$

contain the interaction with the gravitational wave. The physical meaning of $|a_k(t)|^2$ is the probability for finding the perturbed oscillator in the state $|k\rangle$ at the time t. For solving Eq. (5.3) we use an iteration method starting from the initial state, that the oscillator is at the time $t = 0$ in the state $|n\rangle$ $[a_l(t = 0) = \delta_{ln}]$. Herewith we get from (5.3) by a time integration in the *first* iteration step

$$\overset{(1)}{a}_{kn}(t) = -\frac{i}{\hbar} \int_0^t W_{kn}(t')\, e^{i(E_k - E_n)t'/\hbar}\, dt' \tag{5.4}$$

to which we will restrict ourselves in the following. Then the transition probability from the state $|n\rangle$ into the state $|k\rangle$ is given by

$$W_{n \to k} = (1/t)\, |\overset{(1)}{a}_{kn}(t)|^2 \tag{5.5}$$

5.1. *Transition Probabilities*

In view of the matrix elements (5.3a) of the interaction operator W and with respect to the wave metric (3.8) the linear oscillator will be excited only if its linear extension has a component orthogonal to the direction of the wave propagation, which we have chosen as the direction of the x^3 coordinate. Therefore the oscillator shall lie in the x^1, x^2 plane at $x^3 = 0$.

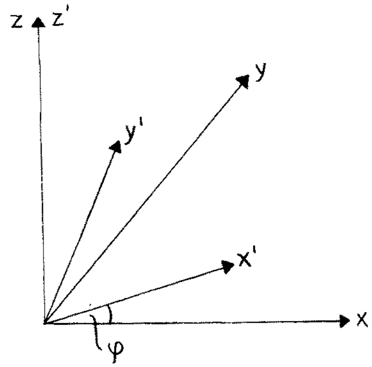

Figure 1—Wave coordinate system $x^1 = x, x^2 = y, x^3 = z$, and oscillator coordinate system $x^{1'} = x', x^{2'} = y', x^{3'} = z'$. The wave propagates along the z axis. The oscillator lies on the x' axis at the origin.

The coordinate in the direction of the oscillator may be signified with $x^{1'} = x'$, whereby the angle between the $x^{1'}$ and the x^1 axis is called φ, see Fig. 1. Furthermore, we restrict ourselves to *linearly* polarized gravitational radiation, consisting of a superposition of compactly neighboring monochromatic components, which is possible in the linear approximation of Einstein's theory. Specifying (3.8) in this sense we get for the wave metric at the position of the oscillator ($x^3 = 0$)

$$h^{ab} = E^{ab} \sum_j \tfrac{1}{2} A_j (e^{-i\omega_j t - i\alpha_j} + e^{i\omega_j t + i\alpha_j})$$
$$E^{11} = -E^{22} = 1, E^{ab} = 0 \text{ otherwise} \tag{5.6}$$

where A_j represents the amplitude and α_j the (arbitrary) phase of the monochromatic wave with the frequency ω_j. Accordingly the angle φ (Fig. 1) means the angle between the polarization of the gravitational wave described by the trace free tensor E^{ab} and the direction of the oscillator.[5]

For calculation of the matrix elements (5.3a) and the expansion coefficients (5.4) it is suitable to use the coordinate $x^{1'} = x'$ of the oscillator as independent variable. Then in view of (5.3a) only the knowledge of $h^{1'1'}$ is necessary because the eigenfunctions φ_l of the undisturbed linear oscillator are dependent on the variable x' alone. From (5.6) we obtain immediately at the position of the oscillator

$$h^{1'1'} = h^{11} \cos 2\varphi = \tfrac{1}{2} \cos 2\varphi \sum_j A_j (e^{-i\omega_j t - i\alpha_j} + e^{i\omega_j t + i\alpha_j}) \tag{5.6a}$$

[5] The linear dimension of the oscillator shall be considered small against the wavelength of the radiation.

and Eq. (5.4) reads, in view of (5.3a)

$$\overset{(1)}{a}_{kn}(t) = -\frac{i\hbar}{4m}\cos 2\varphi \int \varphi_k^* \frac{\partial^2}{\partial x'^2}\varphi_n\, dx'$$

$$\times \sum_j A_j \int_0^t [e^{i(\omega_{kn}-\omega_j)t' - i\alpha_j} + e^{i(\omega_{kn}+\omega_j)t' + i\alpha_j}]\; dt' \tag{5.7}$$

with the abbreviation

$$\omega_{kn} = (E_k - E_n)/\hbar \tag{5.7a}$$

At first we perform the time integration in (5.7) and obtain

$$\overset{(1)}{a}_{kn}(t) = -\frac{\hbar}{4m}\cos 2\varphi \int \varphi_k^* \frac{\partial^2}{\partial x'^2}\varphi_n\, dx'$$

$$\times \sum_j \left\{\frac{A_j e^{-i\alpha_j}}{\omega_{kn}-\omega_j}[e^{i(\omega_{kn}-\omega_j)t} - 1] + \frac{A_j e^{i\alpha_j}}{\omega_{kn}+\omega_j}[e^{i(\omega_{kn}+\omega_j)t} - 1]\right\} \tag{5.8}$$

The first term of the sum contributes a large amount only in case of $\omega_j = \omega_{kn} = (E_k - E_n)/\hbar$ and describes the *absorption* process, through which the oscillator goes over from the energy state E_n into the energy state $E_k > E_n$. On the other hand the second term of the sum becomes very large in case of $\omega_j = -\omega_{kn} = (E_n - E_k)/\hbar$ and represents the *induced emission* process, through which the oscillator goes over from the energy state E_n into the energy state $E_k < E_n$. Because of the symmetry of the expressions for both processes we can for their calculation restrict ourselves to the first one. Then we obtain for the transition probability of the absorption according to (5.5) and (5.8):

$$W_{n\to k} = \frac{\hbar^2}{16m^2 t}\cos^2 2\varphi \left| \int \varphi_k^* \frac{\partial^2}{\partial x'^2}\varphi_n\, dx' \right|^2$$

$$\times \sum_{j,l} \frac{A_j A_l\, e^{-i(\alpha_j - \alpha_l)}}{(\omega_{kn}-\omega_j)(\omega_{kn}-\omega_l)}[e^{i(\omega_{kn}-\omega_j)t} - 1]\,[e^{-i(\omega_{kn}-\omega_l)t} - 1] \tag{5.9}$$

In this double sum only the terms with $\omega_j = \omega_l = \omega_{kn}$, that means the terms $j = l$, contribute the largest amounts, so that (5.9) can be written in good approximation

$$W_{n\to k} = \frac{\hbar^2}{4m^2 t}\cos^2 2\varphi \left| \int \varphi_k^* \frac{\partial^2}{\partial x'^2}\varphi_n\, dx' \right|^2$$

$$\times \sum_j |A_j|^2 \frac{\sin^2 \frac{1}{2}(\omega_{kn}-\omega_j)t}{(\omega_{kn}-\omega_j)^2} \tag{5.10}$$

Subsequently for calculation of the sum in (5.10) we substitute it by an integral according to:

$$\sum_j |A_j|^2 \frac{\sin^2 \frac{1}{2}(\omega_{kn} - \omega_j)t}{(\omega_{kn} - \omega_j)^2} \to \int_0^\infty |A(\omega)|^2 \frac{\sin^2 \frac{1}{2}(\omega_{kn} - \omega)t}{(\omega_{kn} - \omega)^2} d\omega \quad (5.11)$$

wherein $|A(\omega)|^2$ means the spectral intensity of the gravitational wave, [cf. Eq. (5.18)]. The integrand of (5.11) possesses a high maximum at $\omega = \omega_{kn}$ the neighborhood of which contributes large amounts to the integral only, so that under the condition of slow variability of $|A(\omega)|^2$ in the range of this maximum the integral (5.11) can be estimated as follows:

$$\int_0^\infty |A(\omega)|^2 \frac{\sin^2 \frac{1}{2}(\omega_{kn} - \omega)t}{(\omega_{kn} - \omega)^2} d\omega$$

$$\simeq |A(\omega_{kn})|^2 \int_0^\infty \frac{\sin^2 \frac{1}{2}(\omega_{kn} - \omega)t}{(\omega_{kn} - \omega)^2} d\omega = \frac{\pi}{2} t\, |A(\omega_{kn})|^2 \quad (5.12)$$

Herewith Eq. (5.10) takes the form

$$W_{n\to k} = \frac{\pi\hbar^2}{8m^2} \cos^2 2\varphi\, |A(\omega_{kn})|^2 \left| \int \varphi_k^* \frac{\partial^2}{\partial x'^2} \varphi_n \, dx' \right|^2 \quad (5.13)$$

The same expression will be found taking into account the second term of the sum in (5.8). Therefore $W_{n\to k}$, Eq. (5.13), is the transition probability firstly for absorption of gravitational waves by an oscillator in the case of $E_k > E_n$ and secondly for induced emission of gravitational waves in the case of $E_k < E_n$.

Furthermore the transition probabilities for the absorption $n \to k$ and the induced emission of gravitational waves $k \to n$ are equal as can be shown by a transcription of the matrix element in (5.13). Starting from the undisturbed time-independent Schrödinger equation of the linear harmonic oscillator [cf. Eq. (5.1)]

$$\frac{\partial^2}{\partial x'^2} \varphi_n + \frac{2m}{\hbar^2} \left(E_n - \frac{m}{2} \omega_0{}^2 x'^2 \right) \varphi_n = 0 \quad (5.14)$$

wherein ω_0 is the eigenfrequency of the oscillator, we obtain with the help of the orthogonality of the eigenfunctions φ_n

$$\int \varphi_k^* \frac{\partial^2}{\partial x'^2} \varphi_n \, dx' = \left(\frac{m\omega_0}{\hbar} \right)^2 \int \varphi_k^* x'^2 \varphi_n \, dx', \qquad k \neq n \quad (5.14a)$$

Using the mass-quadrupole tensor of the oscillator

$$Q_{a'b'} = \tfrac{1}{2} m (3 x_{a'} x_{b'} - r'^2 \eta_{a'b'}) \quad (5.14b)$$

we find from (5.14a) the relation ($k \neq n$)

$$\int \varphi_k^* \frac{\partial^2}{\partial x'^2} \varphi_n \, dx' = \frac{m\omega_0^2}{\hbar^2} \int \varphi_k^* Q_{1'1'} \varphi_n \, dx' \tag{5.15}$$

and herewith Eq. (5.13) results in

$$W_{n \to k} = \frac{\pi \omega_0^4}{8\hbar^2} \cos^2 2\varphi \, |A(\omega_{kn})|^2 \left| \int \varphi_k^* Q_{1'1'} \varphi_n \, dx' \right|^2 \tag{5.16}$$

Evidently, the square of the matrix element in (5.16) is invariant against changing k and n, so that the transition probabilities for absorption $n \to k$ and induced emission $k \to n$ are equal:

$$W_{n \to k} = W_{k \to n} \tag{5.16a}$$

Moreover, the transition probabilities $W_{n \rightleftarrows k}$ are determined by the product between the square of the matrix element of the mass-quadrupole of the oscillator and the polarization of the gravitational wave expressed by the angle φ. Accordingly Eq. (5.16) can be written with respect to (5.6) and (5.14b) in the general form

$$W_{n \rightleftarrows k} = \frac{\pi \omega_0^4}{18\hbar^2} |A(\omega_{kn})|^2 \left| E^{ab} \int \varphi_k^* Q_{ab} \varphi_n \, d^3x \right|^2 \tag{5.17}$$

where the undashed primordial coordinates are used (see Fig. 1). Of course the transition to any other coordinates is possible because of the scalar character of (5.17).

On the other hand we note that the square of the spectral amplitude of the gravitational wave in (5.13), (5.16), and (5.17) can be expressed by the spectral energy density of the radiation. One finds with the use of the Landau-Lifshitz energy pseudotensor for the metric (5.6) as *mean* spectral energy density

$$\rho(\omega_{kn}) = (\omega_{kn}^2 c^2 / 32\pi f) \, |A(\omega_{kn})|^2 \tag{5.18}$$

(f is Newton's gravitational constant). Similarly to the electromagnetic case Einstein's transition probabilities $B_{n \rightleftarrows k}$ for absorption and induced emission of gravitational waves can be introduced according to

$$W_{n \rightleftarrows k} = \rho B_{n \rightleftarrows k} \tag{5.19}$$

where, with respect to (5.17) and (5.18)

$$B_{n \rightleftarrows k} = \frac{16\pi^2}{9} \frac{\omega_0^4 f}{\hbar^2 \omega_{kn}^2 c^2} \left| E^{ab} \int \varphi_k^* Q_{ab} \varphi_n \, d^3x \right|^2 \tag{5.19a}$$

5.2. Selection Rules

For determination of the selection rules we calculate the matrix elements (5.15), wherein the eigenfunctions φ_n of the oscillator are given by the well-known relations

$$\varphi_n = \left(\frac{m\omega_0}{\hbar\pi}\right)^{1/4} \frac{v_n(\xi)}{(2^n n!)^{1/2}} \tag{5.20}$$

with

$$v_n(\xi) = e^{-\xi^2/2} H_n(\xi), \qquad \xi = x'\sqrt{m\omega_0/\hbar} \tag{5.20a}$$

(H_n are Hermite polynomials). For the functions $v_n(\xi)$ there holds the recursion formula

$$\frac{\partial^2}{\partial\xi^2} v_n = \tfrac{1}{4} v_{n+2} - (n + \tfrac{1}{2})\, v_n + n(n-1)\, v_{n-2} \tag{5.21a}$$

and the relation of orthogonality

$$\int_{-\infty}^{+\infty} v_m(\xi)\, v_n(\xi)\, d\xi = 2^n n!\, \pi^{1/2} \delta_{mn} \tag{5.21b}$$

Herewith the matrix elements (5.15) take the form

$$\int \varphi_k^* \frac{\partial^2}{\partial x'^2} \varphi_n\, dx' = \frac{m\omega_0}{\hbar} (2^{k+n} k!\, n!\, \pi)^{-1/2} \int v_k \frac{\partial^2}{\partial \xi^2} v_n\, d\xi$$

$$= \frac{m\omega_0}{2\hbar} \times \begin{cases} [(n+1)(n+2)]^{1/2} & \text{for } k = n+2 \\ (-2)(n+\tfrac{1}{2}) & \text{for } k = n \\ [n(n-1)]^{1/2} & \text{for } k = n-2 \\ 0 & \text{otherwise} \end{cases} \tag{5.22}$$

Accordingly for absorption and induced emission of gravitational waves by an oscillator there exists the selection rule

$$\Delta n = \pm 2 \tag{5.23}$$

Therefore it follows from the energy eigenvalues $E_n = \hbar\omega_0(n + \frac{1}{2})$ for the absorbed or emitted frequency ω_{kn} according to (5.7a):

$$|\omega_{kn}| = 2\omega_0 \tag{5.24}$$

With the relations (5.22) and (5.24) the transition probabilities for absorption and induced emission of gravitational waves by an oscillator can be written using (5.13) or (5.16) [cf. (5.15) and (5.16a]):

$$W_{n \rightleftarrows n+2} = (\pi/32)\, \omega_0^2 \cos^2 2\varphi\, |A(2\omega_0)|^2 (n+1)(n+2) \tag{5.25}$$

The absorbed or emitted energy per second is given after multiplication with $2\hbar\omega_0$ by

$$L_{n \rightleftarrows n+2} = (\pi/16)\, \hbar\omega_0^3 \cos^2 2\varphi\, |A(2\omega_0)|^2 (n+1)(n+2) \tag{5.26}$$

Evidently for $n \gg 1$ the power $L_{n \rightleftarrows n+2}$ of the oscillator increases quadratically with the quantum number n.

5.3. *Effective Absorption Power of Oscillators*

The last result suggests, that the absorption power of oscillators increases very rapidly for higher excitation states. However, it must be taken into account that not only do oscillators in the state $|n\rangle$ absorb energy going over into the state $|n+2\rangle$, but also energy will be emitted by the oscillators through transition into the state $|n-2\rangle$. Therefore the *effective* absorption power of an oscillator in the state $|n\rangle$ is given by (compare van Vleck, 1924)

$$L_n = 2\hbar\omega_0(W_{n\to n+2} - W_{n\to n-2}) \tag{5.27}$$

Insertion of (5.25) results in (with respect to (5.18) and (5.24))

$$\begin{aligned} L_n &= \tfrac{1}{4}\pi\hbar\omega_0{}^3 \cos^2 2\varphi\,|A(2\omega_0)|^2\,(n+\tfrac{1}{2}) \\ &= 2\pi^2 f\hbar\omega_0 \cos^2 2\varphi(n+\tfrac{1}{2})\,\rho(2\omega_0)/c^2 \end{aligned} \tag{5.28}$$

Accordingly the *effective* absorption power of gravitational waves by oscillators increases *linearly* with the quantum number n. Thus oscillators in highly excited states seem to be more appropriate for detection of gravitational waves than oscillators in the ground state.

It should be pointed out that, in contrast to Eq. (5.26), the result (5.28) allows the transition to the classical theory in case of $n \gg 1$, taking into consideration that then, as is well known,

$$l = \sqrt{2\hbar n/\omega_0 m} \tag{5.29}$$

represents the amplitude of the classical linear oscillator with the energy E_n. Thus one finds from (5.28) in the classical limit the following effective absorption power for oscillators:

$$\begin{aligned} L &= \tfrac{1}{8}\pi\omega_0{}^4 \cos^2 2\varphi\,|A(2\omega_0)|^2\,ml^2 \\ &= \pi^2 f\omega_0{}^2 \cos^2 2\varphi\, ml^2 \rho(2\omega_0)/c^2 \end{aligned} \tag{5.30}$$

in accordance with the classical result of Misner (et al.) (1973), whereby ml^2 is an immediate measure for the mass-quadrupole moment of the oscillator.[6,7]

6. *Spontaneous Emission of Gravitational Radiation*

With the knowledge of the transition probabilities for absorption and induced emission it is possible to calculate the transition probabilities of the spontaneous emission of gravitational radiation $A_{n\to k}$ using the connection

$$A_{n\to k} = (\hbar\omega_{kn}^3/8\pi^3c^3)\,B_{n\to k} \tag{6.1}$$

[6] It should be remarked, that the classical results deduced in this paper are based on the correspondence relation (5.29).

[7] The quotient $L/\rho c$ according to (5.28) and (5.30) represents the effective cross section of the quantum mechanical and the classical oscillator respectively.

derived first by Einstein (1917) (see also, e.g., Blochinzew, 1972). Insertion of (5.19a) gives with respect to (5.24)

$$A_{n\to k} = \frac{|\omega_{kn}|^5 f}{72\pi\hbar c^5} |E^{ab} \int \varphi_k^* Q_{ab}\varphi_n \, d^3x|^2 \tag{6.2}$$

which represents the probability for spontaneous emission of a graviton per second and solid angle element with a polarization described by the tensor E^{ab}.

Finally we calculate the total gravitational energy emitted by an oscillator using the dashed coordinates (oscillator system, see Fig. 1). Then the quadrupole tensor of the linear oscillator obeys the relation

$$Q_{1'1'} = -2Q_{2'2'} = -2Q_{3'3'}, \qquad Q_{a'b'} = 0 \qquad \text{for } a' \neq b' \tag{6.3}$$

and therefore in view of Eq. (6.2) only the diagonal elements of the polarization tensor $E^{a'b'}$ are to be determined for all radiation directions with the angle ϑ against the oscillator axis ($x^{1'}$-axis). One gets

$$\begin{aligned} E^{1'1'} &= E^{11} \sin^2 \vartheta \\ E^{2'2'} &= E^{11}(\cos^2 \chi \cos^2 \vartheta - \sin^2 \chi) \\ E^{3'3'} &= E^{11}(\sin^2 \chi \cos^2 \vartheta - \cos^2 \chi) \end{aligned} \tag{6.4}$$

wherein $E^{11} = -E^{22} = 1$ are the only nonvanishing components of the polarization tensor in the coordinate system determined by the radiation direction (x^3 axis) and χ is the rotational angle around the $x^{1'}$-axis (see Fig. 2). With (6.3) and (6.4) we obtain from (6.2) by multiplication with $\hbar|\omega_{kn}|$ the following mean energy loss radiated into the solid angle element $d\Omega$:

$$d\left(\frac{dE}{dt}\right)_{n\to k} = -\frac{\omega_{kn}^6 f}{32\pi c^5} \sin^4 \vartheta \, |\int \varphi_k^* Q_{1'1'}\varphi_n \, dx'|^2 \, d\Omega \tag{6.5}$$

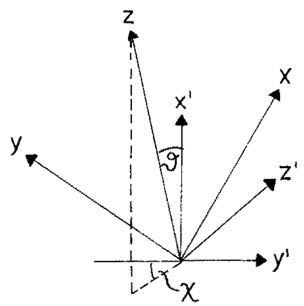

Figure 2–Orientation of the oscillator system (x', y', z') and the wave system (x, y, z). The z, x', and x axes lie in the same plane in view of the linear polarization of the gravitational wave emitted by the oscillator (x' axis) in the direction of the z axis.

assuming that the wavelength of the gravitational radiation $\lambda = \pi c/\omega_0$ is large against the projection of the linear dimension of the oscillator given by (5.29) in the direction of the wave. Integration over the total solid angle yields

$$\left(\frac{dE}{dt}\right)_{n\to k} = -\frac{\omega_{kn}^6 f}{15c^5} \left| \int \varphi_k^* Q_{1'1'} \varphi_n \, dx' \right|^2 \tag{6.6}$$

This total mean energy loss per second corresponds exactly to the classical result regarding Eq. (5.24) for the emitted frequency ($|\omega_{kn}| = 2\omega_0$) and substituting the matrix element of the quadrupole operator by the mean quadrupole tensor itself (compare Rosen & Shamir, 1957; Landau & Lifshitz, 1967).

With the relation (5.15), the calculation of the matrix elements (5.22), and the emitted frequency (5.24) we get from (6.6) immediately as energy loss by spontaneous emission from the state $|n\rangle$

$$\left(\frac{dE}{dt}\right)_n = -\frac{16 f \hbar^2 \omega_0^4}{15c^5} n(n-1) \tag{6.7}$$

Accordingly the gravitational radiation energy emitted by an oscillator increases very rapidly for higher excited states ($\sim n^2$). For the two states $n = 0$ *and* $n = 1$ there exists no emission of gravitational radiation because of the selection rule $\Delta n = \pm 2$ [compare (5.23)].

In the classical limit ($n \gg 1$), it follows from (6.7) with the use of (5.29)

$$\frac{dE}{dt} = -\frac{4}{15} \frac{f \omega_0^6}{c^5} (ml^2)^2 \tag{6.8}$$

wherein ml^2 represents, as in Eq. (5.30), a measure for the mass-quadrupole moment of the oscillator. This result for the mean energy loss coincides with that of the classical theory in consequence of the exact correspondence between the quantum mechanical formula (6.6) and the classical expression.

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